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Some results related to constrained non-differentiable (non-smooth) pseudolinear minimization problems

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Abstract: This paper deals with the minimization of a class of non-differentiable (non-smooth) pseudolinear functions over a closed and convex set subject to linear inequality constraints. The properties of locally Lipschitz pseudolinear functions are used to establish several Lagrange multiplier characterizations of the solution set of the minimization problem. We derive certain conditions, under which an efficient solution becomes a properly efficient solution of a constrained non-differentiable minimization problem.

Keywords: Efficient solutions, Locally Lipschitz functions, Properly efficient solutions, Pseudolinear functions, Solution sets

INTRODUCTION

The characterizations and the properties of the solution set related to an optimization problem having multiple optimal solutions are of fundamental importance in understanding the behavior of solution methods. Mangasarian [1] has given simple and elegant characterizations for the solution set of convex extremum problems with one solution known. These results have been further extended to various classes of optimization problems infinite dimensional convex optimization problems [2, 3], generalized convex optimization problems [4-7] and convex vector optimization problems [8]. Mangasarian [9] has introduced the concept of pseudoconvex and pseudoconcave functions as generalization of convex and concave functions respectively. Chew et al. [10] have introduced pseudolinear functions. First and second order characterizations od differentiable pseudolinear functions have been obtained in [10, 11]. Jeyakumar et al. [6] have obtained the characterizations of the solution set of differentiable pseudolinear program. Lu et al. [12] have established some characterizations for locally Lipschitz pseudolinear functions and solution set of a pseudolinear program using Clarke subdifferential on Banach spaces. Dinh et al. [13] have shown several Lagrange multiplier characterizations of a pseudolinear optimization problem over a closed convex set with linear inequality constraints.

This paper has different sections. Section 2 consists some basic definitions and preliminary results. In section 3, we establish Lagrange multiplier characterizations of the solution set of a constrained non-smooth pseudolinear optimization problem with a linear inequality constraints in terms of Clarke subdifferential. In section 4, we derive certain conditions, under which an efficient solution becomes a properly efficient solution of a constrained non-smooth (non-differentiable) vector pseudolinear minimization problem. Section 5 consists conclusions on our results.

Definitions and Preliminaries

Let R^k be the k-dimensional Euclidean space and R_+ be the positive orthant of R. Let S be any non-empty subset of R^k and $\langle .,. \rangle$ denote the Euclidean inner product.

The following definitions and Lemmas are from [15].

Definition 2.1 A function $g : S \rightarrow R$ is said to be locally Lipschitz at $s \in S$, if and only if there exists a positive number L and a neighborhood N of s such that, for any $x, y \in N$, one has

$|g(x) - g(y)| \leq L\|x - y\|$.

The function g is said to be Lipschitz on S, if and only if the above condition is satisfied for all $s \in S$.

Definition 2.2 Let $g : S \rightarrow R$ be a locally Lipschitz function at $s \in S$. The Clarke generalized directional derivative of g at $s \in S$ in the direction of vector $v \in R^k$ is denoted by $g^o(s\,; v)$ and is defined as

$$g^o(s, v) = \lim\ \sup_{\substack{y \to s \\ t \downarrow 0}} \frac{g(y+tv) - g(y)}{t}.$$

Definition 2.3 Let $g : S \rightarrow R$ be a locally Lipschitz function at $s \in S$. The Clarke generalized subdifferential of g at $s \in S$ is denoted by $\partial^c g(s)$ and is defined as

$$\partial^c g(s) = \{\varphi \in R^k :\ g^o(s, v) \geq (\varphi, v), \forall\, v \in R^k\}.$$

Definition 2.4 Let M be any non-empty and closed subset of R^k. The Clarke tangent cone of M at s is denoted by $T_M(s)$ and is defined as
$T_M(s) = \{t \in R^k : d_M^o(s,t) = 0\}$,
Where d_M denotes the distance function related to M and
$d_M(s) = \inf\{\|s - t\| : t \in M\}$.
Definition 2.5 Let M be any non-empty and closed subset of R^k. The Clarke normal cone of M at s is denoted by $N_M(s)$ and is defined as
$N_M(s) = \{\varphi \in R^k : \langle \varphi, v\rangle \leqq 0, \forall\, v \in T_M(s)\}$.
Lemma 2.1 Let $g : S \to R$ be a locally Lipschitz function with Lipschitz constant L at $s \in S$. Then $\partial^c g(s)$ is a non-empty convex and compact subset of R^k and $\|\varphi\| \le L, \forall\, \varphi \in \partial^c g(s)$.
Lemma 2.2 (Lebourg mean value theorem) Let $s, t \in S$ and suppose that $g : S \to R$ be locally Lipschitz on an open set including the line segment $[s, t]$. Then there exists a point u in $]s, t[$ such that
$g(s) - g(t) \in \langle \partial^c g(u), s - t\rangle$,
where $]s, t[$ denotes the line segment joining s and t not including end points s and t.
Lemma 2.3 Let $g : S \to R$ be a locally Lipschitz function at $s \in S$. If $\{s_k\}$ and $\{\varphi_k\}$ are two sequences in R^k such that $\varphi_k \in \partial^c g(s_k)$ for all k and if $s_k \to s$ and φ is a cluster point of $\{\varphi_k\}$, then $\varphi \in \partial^c g(s)$.
Definition 2.6 ([12, 16]) Let S be an open convex set of R^k. A locally Lipschitz function $g : S \to R$ is said to be pseudoconvex on S, if and only if for all $s, t \in S$, one has

$$g(t) < g(s) \Longrightarrow \langle \varphi, t - s\rangle < 0, \forall\, \varphi \in \partial^c g(s)$$

or equivalently
there exists $\varphi \in \partial^c g(s) : \langle \varphi, t - s\rangle \ge 0 \implies g(t) \ge g(s)$.
Definition 2.7 Let S be an open convex set of R^k. A locally Lipschitz function $g : S \to R$ is said to be pseudoconcave on S, if and only if $(-g)$ is pseudoconvex on S.
Definition 2.8 Let S be an open convex set of R^k. A locally Lipschitz function $g : S \to R$ is said to be pseudolinear on S, if and only if g is both pseudoconvex and pseudoconcave on S.
Definition 2.9 ([2]) Let C be a non-empty convex set of R^k. The relative interior of C, denoted by ri C, is the set of all those $c \in C$, for which cone $(C - c)$ is a subspace.
Definition 2.10 ([18]) Let B be a closed and convex set of R^k. The recession cone of B is denoted by B^∞ and is defined as

$$B^\infty = \{x \in R^k : b + tx \in B, \forall\, t \geqq 0, \forall b \in B\}.$$

The set B is bounded if and only if $B^\infty = \{0\}$
Lemma 2.4 Let S be an open convex set of R^k. If $g : S \to R$ be a locally Lipschitz pseudolinear function on S, then $g(s) = g(t)$ if and only if there exists $\varphi \in \partial^c g(s)$ such that $\langle \varphi, t - s\rangle = 0$, $\forall\, s, t \in S$.
Lemma 2.5 Let S be an open convex set of R^k and $g : S \to R$ be a locally Lipschitz function on S. Then, the function g is pseudolinear on S, if and only if there exists a function $p : S \times S \to R_+$ such that, for every $s, t \in S$ there exists $\varphi \in \partial^c g(s)$ such that $g(t) = g(s) + p\langle s, t\rangle\langle \varphi, t - s\rangle$

Proposition 2.1 Let S be an open convex set of R^k and $g : S \to R$ be a locally Lipschitz pseudolinear function on S. Let $u \in S$, then the set
$T = \{s \in S : g(s) = g(u)\}$ is convex.
Proof Let $t, u \in T$, then we have $g(s) = g(t) = g(u)$
By Lemma 2.4, there exists $\varphi \in \partial^c g(s)$ such that $\langle \varphi, t - s\rangle = 0$.
Then, for every $\lambda \in [0,1]$, $\exists\, \varphi \in \partial^c g(s)$ such that

$$\langle \varphi, ((1 - \lambda)s + \lambda t) - s\rangle = \lambda\langle \varphi, t - s\rangle = 0$$

Thus, using Lemma 2.4, again we infer that

$$g((1 - \lambda)s + \lambda t) = g(s) = g(u), \qquad \forall\, \lambda \in [0,1].$$

Hence, $(1 - \lambda)s + \lambda t \in T$, $\forall\, \lambda \in [0,1]$.
Therefore, T is convex.

Proposition 2.2 Let S be an open convex set of R^k and $g : S \to R$ be a locally Lipschitz pseudolinear function on S. Let $s \in S$ be arbitrary, then for any $\varphi, \varphi' \in \partial^c g(s)$ there exists $\lambda > 0$ such that $\varphi' = \lambda\varphi$.

Non-differentiable (non-smooth) Pseudolinear Optimization Problems
In this section, we derive Lagrange multiplier characterizations of a non-smooth pseudolinear optimization problem with a linear inequality constraints in terms of Clarke subdifferential.
We consider the following constrained non-smooth pseudolinear optimization problem (NPOP):
(NPOP) Minimize $g(s)$

subject to $s \in D = \{s \in C: a_j^T s \leqq b_j,\ j = 1,2,3,\dots\dots,m\}$,

where $g : S \to R$ is a locally Lipschitz pseudolinear function and D is contained in an open convex set S, $a_j \in R^k, b_j \in R$ and C is a closed and convex subset of R^k.

Let $U = \{s \in D: g(s) \leqq g(t), \forall t \in D\}$ (1)

The set U is the solution set of the non-smooth pseudolinear optimization problem (NPOP). The set U is non-empty convex subset of R^k.

Now, we state the following necessary and sufficient optimality conditions which are non-smooth version of the result given in [13].

Proposition 3.1 Let $s \in D$ and let the (NPOP) satisfies some suitable constraint qualification such as, there exists $s_0 \in \text{ri } C$, such that $a_j s_0 \leqq b_j$ for all $j \in J$, where ri C is the relative interior of the set C. Then, $s \in U$ if and only if there exists a Lagrange multiplier $\lambda = (\lambda_1, \lambda_2, \lambda_3, \dots\dots\dots, \lambda_m) \in R^m$ such that

$$0 \in \partial^c g(s) + \sum_{j \in J} \lambda_j a_j + N_C(s),$$

$$\lambda_j \geqq 0, \quad \lambda_j(a_j^T s - b_j) = 0, \quad \forall j \in J \tag{2}$$

where $N_C(s)$ is the Clarke normal cone of C at s.

The following theorem establishes that the active constraints corresponding to a known solution of the (NPOP) remain active at all the solutions of the (NPOP).

Theorem 3.1 Let $s \in U$ and let the (NPOP) satisfies optimality condition (2) with a Lagrange multiplier $\lambda = (\lambda_1, \lambda_2, \lambda_3, \dots\dots\dots, \lambda_m) \in R^m$. Then, for each $t \in U$, $\sum_{j \in J(s)} \lambda_j(a_j^T t - b_j) = 0$ and $g(\cdot) + \sum_{j \in J(s)} \lambda_j(a_j(\cdot) - b_j)$ is a constant function on U.

Proof Using (2), there exists $\varphi \in \partial^c g(s)$ such that

$$\langle \varphi, t - s \rangle + \sum_{j \in J(s)} \lambda_j a_j^T (t - s) \geqq 0, \qquad \forall\, t \in C. \tag{3}$$

Since, g is a locally Lipschitz pseudolinear function on S, by Lemma 2.5, there exists a positive real valued function $p: S \times S \to R_+$ and $\varphi' \in \partial^c g(s)$ such that

$g(t) = g(s) + p\langle s, t\rangle \langle \varphi', t - s\rangle$. (4)

Using Proposition 2.2, there exists $\mu > 0$ such that $\varphi' = \mu\varphi$.

Hence from (4), we have

$$g(t) = g(s) + \mu p\langle s, t\rangle \langle \varphi, t - s\rangle.$$

Using (2), (3) and (4), we have

$$g(t) - g(s) + \mu p\langle s, t\rangle \sum_{j \in J(s)} \lambda_j \left(a_j^T t - b_j\right) \geqq 0, \qquad \forall\, t \in C.$$

Since for each $t \in U$, $g(t) = g(s)$, it follows that

$$\mu p\langle s, t\rangle \sum_{j \in J(s)} \lambda_j \left(a_j^T t - b_j\right) \geqq 0$$

Since $p\langle s, t\rangle > 0$ and $\mu > 0$, we have

$$\sum_{j \in J(s)} \lambda_j \left(a_j^T t - b_j\right) \geqq 0.$$

From the feasibility of t, we have

$$\sum_{j \in J(s)} \lambda_j \left(a_j^T t - b_j\right) \leqq 0.$$

Hence, we get

$$\sum_{j \in J(s)} \lambda_j \left(a_j^T t - b_j\right) = 0.$$

which proves that $g(\cdot) + \sum_{j \in J(s)} \lambda_j(a_j(\cdot) - b_j)$ is a constant function on U.

Now, we assume that $s \in U$ and $\lambda \in R^m$ is a Lagrange multiplier corresponding to s. Let $\bar{J}(s) = \{j \in J(s): \lambda_j > 0\}$ and $t\alpha s = (1 - \alpha)t + \alpha s, \alpha \in [0,1]$. Now, we establish the characterization of the solution set of the (NPOP).

Theorem 3.2 Let $s \in U$ and let the (NPOP) satisfies optimality condition (2) with a Lagrange multiplier $\lambda = (\lambda_1, \lambda_2, \lambda_3, \dots\dots\dots, \lambda_m) \in R^m$. Let

$$\bar{U}_1 = \{t \in C:\ a_j^T t = b_j, \forall j \in \bar{J}(s),\ a_j^T t \leqq b_j, \forall j \in J \setminus \bar{J}(s), \quad \exists\, \varphi \in \partial^c g(t): \langle \varphi, t - s\rangle = 0\}$$

$$\bar{U}_2 = \{t \in C:\ a_j^T t = b_j, \forall j \in \bar{J}(s),\ a_j^T t \leqq b_j, \forall j \in J \setminus \bar{J}(s), \quad \exists\, \Psi \in \partial^c g(t): \langle \Psi, t - s\rangle = 0\}$$

$$\overline{U}_3 = \{t \in C:\ a_j^T t = b_j, \forall j \in \overline{J}(s),\ a_j^T t \leqq b_j, \forall j \in J \setminus \overline{J}(s), \quad \exists\, \xi \in \partial^c g(t\alpha s): \langle \xi, t - s\rangle = 0, \forall \alpha \in [0,1]\}$$

Then, $U = \overline{U}_1 = \overline{U}_2 = \overline{U}_3$.

Proof We prove the equality $U = \overline{U}_2$.

Let $s \in U$, then $g(s) = g(t)$

By Theorem 3.1, we have

$$\sum_{j \in J(t)} \lambda_j \left(a_j^T s - b_j\right) = 0$$

Therefore, $a_j^T s = b_j$, $\forall j \in J \setminus \overline{J}(t)$

Again, since $g(s) = g(t)$, using Lemma 2.4, $\exists\, \Psi \in \partial^c g(t)$ such that $\langle \Psi, s - t\rangle = 0$, which provides $s \in \overline{U}_2$.

Thus, $U \subseteq \overline{U}_2$.

Conversely, let $s \in \overline{U}_2$, then $s \in C$, $a_j^T s \leqq b_j, \forall j \in J$

$\exists\, \Psi \in \partial^c g(t)$ such that $\langle \Psi, s - t\rangle = 0$

By Lemma 2.4, $s \in D$ and $g(s) = g(t)$, which implies that $s \in U$.

Thus, $\overline{U}_2 \subseteq U$.

Therefore, $U = \overline{U}_2$.

Similarly, we can prove other equalities.

Hence, $U = \overline{U}_1 = \overline{U}_2 = \overline{U}_3$.

Corollary 3.1 Let $s \in U$ and let the (NPOP) satisfies optimality condition (2) with a Lagrange multiplier $\lambda = (\lambda_1, \lambda_2, \lambda_3, \dots\dots\dots, \lambda_m) \in R^m$. Let

$$\widetilde{U}_1 = \{t \in C:\ a_j^T t = b_j, \forall j \in \overline{J}(s),\ a_j^T t \leqq b_j, \forall j \in J \setminus \overline{J}(s), \quad \exists\, \varphi \in \partial^c g(t): \langle \varphi, t - s\rangle \leqq 0\}$$

$$\widetilde{U}_2 = \{t \in C:\ a_j^T t = b_j, \forall j \in \overline{J}(s),\ a_j^T t \leqq b_j, \forall j \in J \setminus \overline{J}(s), \quad \exists\, \Psi \in \partial^c g(t): \langle \Psi, t - s\rangle \leqq 0\}$$

$$\widetilde{U}_3 = \{t \in C:\ a_j^T t = b_j, \forall j \in \overline{J}(s),\ a_j^T t \leqq b_j, \forall j \in J \setminus \overline{J}(s), \quad \exists\, \xi \in \partial^c g(t\alpha s): \langle \xi, t - s\rangle \leqq 0, \forall \alpha \in [0,1]\}$$

Then, $U = \widetilde{U}_1 = \widetilde{U}_2 = \widetilde{U}_3$.

Proof Clearly, $\overline{U}_1 \subseteq \widetilde{U}_1, \overline{U}_2 \subseteq \widetilde{U}_2, \overline{U}_3 \subseteq \widetilde{U}_3$.

Let $s \in \widetilde{U}_1$, then $a_j^T s = b_j, \forall\, j \in \overline{J}(s)$ and $a_j^T s \leq b_j, \forall j \in J \setminus \overline{J}(s)$

Using Lemma 2.5, $\exists\, \varphi \in \partial^c g(s)$ such that

$$g(t) - g(s) = p(s,t)\langle \varphi, t - s\rangle \geqq 0$$

which implies that $g(s) \leqq g(t)$.

As $t \in U$, we get $g(s) = g(t)$

Hence, $s \in U$.

Using Lemma 2.5, $\exists\, \Psi \in \partial^c g(s)$ such that

$$g(t) - g(s) = p(s,t)\langle \Psi, t - s\rangle$$

Using above equality, we can prove that $\widetilde{U}_2 \subseteq \overline{U}_2$

Using Lemma 2.5, $\exists\, \xi \in \partial^c g(s)$ such that

$$g(t) - g(t\alpha s) = (1-\alpha)p(t\alpha s, t)\langle \xi, t - s\rangle, \forall\, \alpha \in [0,1]$$

Using above equality, we can prove that $\widetilde{U}_3 \subseteq \overline{U}_3$.

Hence, $U = \widetilde{U}_1 = \widetilde{U}_2 = \widetilde{U}_3$.

Now, we will prove that for the case $C = R^k$, a polyhedral convex subset of R^k is the solution set of the (NPOP). We establish the characterization independently from the objective function g.

Theorem 3.3 Suppose that for $C = R^k$ and $s \in U$, the (NPOP) satisfies optimality condition (2) with a Lagrange multiplier $\lambda = (\lambda_1, \lambda_2, \dots\dots\dots, \lambda_m) \in R^m$. Then,

$U = \{t \in R^k:\ a_j^T t = b_j, \forall j \in \overline{J}(s),\ a_j^T t \leqq b_j, \forall j \in J \setminus \overline{J}(s)\}$.

And, hence, U is a polyhedral convex subset of R^k.

Proof Using (2), there exists $\varphi \in \partial^c g(s)$ such that

$$\langle \varphi, t - s\rangle + \sum_{j \in J(s)} \lambda_j a_j^T (t - s) \geqq 0, \qquad \forall\, t \in R^k. \tag{5}$$

Let $t \in R^k$ satisfies $a_j^T t = b_j, \forall j \in \overline{J}(s)$ and $a_j^T t \leq b_j, \forall j \in J \setminus \overline{J}(s)$, then from (5), there exists $\varphi \in \partial^c g(s)$ such that

$\langle \varphi, t - s\rangle = 0$.

Corollary 3.2 Suppose that for $C = R^k$ and $s \in U$, the (NPOP) satisfies optimality condition (2) with a Lagrange multiplier $\lambda = (\lambda_1, \lambda_2, \dots\dots\dots, \lambda_m) \in R^m$. Then, the solution set U is bounded if and only if

$$\{t \in R^k:\ a_j^T t = 0, \forall j \in \overline{J}(s),\ a_j^T t \leqq 0, \forall j \in J \setminus \overline{J}(s)\} = \{0\}.$$

Proof From Theorem 3.3, we have

$$U^{\infty} = \{t \in R^k:\ a_j^T t = 0, \forall j \in \bar{J}(s),\ a_j^T t \leqq 0, \forall j \in J \backslash \bar{J}(s)\}$$

Where U^{∞} is the recession cone of U. It is obvious by the definition that the set U is bounded if and only if $U^{\infty} = 0$.

4 Non-differentiable (non-smooth) Vector Pseudolinear Optimization Problem (NVPOP) with Linear Inequality Constraints

We consider the following non-smooth vector pseudolinear optimization problem (NVPOP) with linear inequality constraints:

(NVPOP) Minimize $g(s) = (g_1(s), g_2(s), g_3(s), \dots\dots, g_m(s))$

Subject to $s \in \widehat{D} = \{s \in R^k:\ a_j^T s \leq b_j, j \in J = \{1, 2, 3, \dots\dots, n\}\}$

where $a_j \in R^k$, $b_j \in R$

$g_i: S \longrightarrow R,\ i \in I = \{1, 2, 3, \dots\dots, m\}$ are locally Lipschitz pseudolinear functions on open convex set S, containing $\widehat{D}$ with respect to the same proportional function $\tilde{p}$.

Let $J(t) = \{ j \in J:\ a_j^T t = b_j\}$

Definition 4.1 ([19]) A point $s \in \widehat{D}$ is said to be an efficient solution of the (NVPOP), if and only if there exists no $t \in \widehat{D}$ such that

$g_i(t) \leq g_i(s), \quad \forall\, i \in I$

$g_q(t) < g_q(s), \quad$ for some $q \in I$.

Let $\widehat{U}$ be the set of all efficient solution of the (NVPOP) and $\widehat{U} \neq \emptyset$.

Definition 4.2 ([19]) An efficient solution $s \in \widehat{U}$ is said to be properly efficient solution of the (NVPOP), if and only if there exists a scalar $\hat{L} > 0$ such that, for all $i \in I$, the inequality

$\frac{g_i(t)-g_i(s)}{g_q(s)-g_q(t)} \leqq \hat{L}$

holds for some $q \in I$ such that $g_q(t) < g_q(s)$ whenever $s \in \widehat{D}$ and $g_i(t) > g_i(s)$.

In the following theorem, we establish the conditions under which an efficient solution becomes a properly efficient solution.

Theorem 4.1: Let $s \in \widehat{D}$ for the (NVPOP). Then the following statements are equivalent:

(i) s is an efficient solution of the (NVPOP), i.e. $s \in \widehat{U}$,

(ii) $\exists\ \lambda_i > 0,\ i = 1, 2, 3, 4, \dots\dots, m$ such that

$$\sum_{i=1}^{m} \lambda_i\, g_i(t) \geq \sum_{i=1}^{m} \lambda_i\, g_i(s), \quad \forall t \in \widehat{D},$$

(iii) s is a properly efficient solution of the (NVPOP).

Proof : We shall prove the following implications.

Implication (i) $\Longrightarrow$ (ii)

Implication (ii) $\Longrightarrow$ (iii)

Implication (iii) $\Longrightarrow$ (i)

From the definitions (4.1) and (4.2), the implication (iii) $\Longrightarrow$ (i) is obvious.

From [18], we have

The implication (ii) $\Longrightarrow$ (iii)

Now, We shall prove the implication (i) $\Longrightarrow$ (ii).

Let $s \in \widehat{D}$ is an efficient solution of the (NVPOP). Then by Theorem 2.1 in [14], there exist $\lambda_i > 0,\ i = 1, 2, 3, 4, \dots\dots, m$, $\beta_j \geq 0$, $j \in J(s)$ and there exists $\varphi_i \in \partial^c g_i(s)$ such that

$$\sum_{i=1}^{m} \lambda_i\, \varphi_i + \sum_{j \in J(s)} \beta_j\, a_j = 0.$$

Therefore, $\forall\ t \in \widehat{D}$, $\exists\ \varphi_i \in \partial^c g_i(s)$ such that

$$\sum_{i=1}^{m} \lambda_i \langle \varphi_i, t - s\rangle + \sum_{j \in J(s)} \beta_j\, a_j^T(t - s) = 0. \tag{6}$$

Since, for all $j \in J(s)$, $a_j^T s = b_j$, therefore, equation (6) becomes

$$\sum_{i=1}^{m} \lambda_i \langle \varphi_i, t - s\rangle + \sum_{j \in J(s)} \beta_j\, (a_j^T t - b_j) = 0, \qquad \forall t \in \widehat{D}$$

Since $(a_j^T t - b_j) \leq 0$, therefore for any $t \in \widehat{D}$, $\exists\ \varphi_i \in \partial^c g_i(s)$ such that

$$\sum_{i=1}^{m} \lambda_i \langle \varphi_i, t - s \rangle \geq 0. \tag{7}$$

Since $g_i, i = 1, 2, 3, 4, \dots \dots, m$ are pseudolinear functions with respect to $\tilde{p}$, therefore $\exists\ \varphi_i' \in \partial^c g_i(s)$ such that

$$g_i(t) - g_i(s) = \tilde{p}(s,t)\langle \varphi_i', t - s\rangle, \qquad \forall t \in \widehat{D}. \tag{8}$$

From Proposition 2.2, $\exists\ \mu_i > 0, i = 1, 2, 3, 4, \dots \dots \dots, m$ such that $\varphi_i' = \mu_i \varphi_i$.
Therefore equation (8) becomes

$$g_i(t) - g_i(s) = \tilde{p}(s,t)\mu_i\langle \varphi_i, t - s\rangle, \qquad \forall t \in \widehat{D}. \tag{9}$$

Since $\tilde{p}(s,t) > 0$ and $\mu_i > 0, i = 1, 2, 3, 4, \dots \dots \dots, m$
Therefore from equations (7) and (9), we have

$$\sum_{i=1}^{m} \lambda_i\, g_i(t) \geq \sum_{i=1}^{m} \lambda_i\, g_i(s), \quad \forall t \in \widehat{D}.$$

which implies that (ii) holds.
Thus, implication (i) $\Longrightarrow$ (ii). Which completes the proof.

CONCLUSIONS

In this paper, we have studied the minimization (optimization) of a non-differentiable (non-smooth) locally Lipschitz pseudolinear function subject to linear inequality constraints over a closed and convex set. We have used the properties of locally Lipschitz pseudolinear functions to establish several Lagrange multiplier characterizations of the solution set of the minimization (optimization) problem (NPOP). We have shown that the Lagrangian function is consant on the solution set of (NPOP). We have shown that the conditions for an efficient solution of constrained non-differentiable (non-smooth) vector pseudolinear optimization problem (NVPOP) to be a properly efficient solution.

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